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E11 and the non-linear dual graviton

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ABSTRACT

The non-linear dual graviton equation of motion as well as the duality relation between the gravity and dual gravity fields are found in E theory by carrying out E_{11} variations of previously found equations of motion. As a result the equations of motion in E theory have now been found at the full non-linear level up to, and including, level three, which contains the dual graviton field. When truncated to contain fields at levels three and less, and the spacetime is restricted to be the familiar eleven dimensional space time, the equations are equivalent to those of eleven dimensional supergravity.

1. Introduction

It was conjectured that the non-linear realisation of the semi-direct product of E_{11} with its vector representation (l_1), denoted $E_{11} \otimes_s l_1$, leads to a theory, called E theory, that contains the eleven dimensional supergravity theory [1,2]. E theory contains an infinite number of fields that live on a space time that has an infinite number of coordinates. However, the fields at low levels are just those of eleven dimensional supergravity, in particular at levels zero, one and two we find the graviton, three and six form fields respectively. Furthermore, the lowest level coordinate is that of our familiar eleven dimensional space time. Following early efforts, for example reference [3], it has been shown that E theory leads to essentially unique equations of motion which are second order in derivatives for the graviton and form fields [4,5]. When these equations of motion are restricted to contain only the supergravity fields which live on the usual spacetime they are just those of eleven dimensional supergravity [3,4]. By taking different decompositions of E_{11} one can find all the maximal supergravity theories including the gauge supergravities and it is inevitable that the analogous result for the equations of motion holds in all dimensions, for a review and references therein see [6].

The equations of motion at the linearised level have been found in eleven dimensions up to and including level four in the fields [7]. The field at level three is the dual graviton while at level four there are three fields. It was shown that the dual graviton field obeyed an equation that did correctly describe the degrees of freedom of gravity and while one of the fields at level four leads to Romans theory, when dimensional reduced to ten dimensions, one of the other level four fields was dual of the three form. The degrees of freedom resulting from the full system of equations were those of eleven dimensional supergravity.

The dual graviton first appeared in five dimensions in the reference [8]. This paper was one of the first to consider the dynamics of fields which carried mixed symmetry indices and it constructed the equation of motion of the field $\phi_{ab,c}$ with the symmetries $\phi_{ab,c} = -\phi_{ba,c}$ in a general dimension. The author noted that in the massless case, and in five dimensions, the equation of motion for this field had the same number of on-shell degrees of freedom as the graviton. In reference [9] the field $\phi_{a_1 \ldots a_{D-3},b}$ was considered in D dimensions and suggested as a candidate for the dual graviton in the sense that a quantity which contained two space–time derivatives acting on this field could be regarded as being dual to the Riemann tensor. By assuming the existence of an appropriate light cone formulation, which would suitable restrict the indices on the irreducible component of this field to take only $D-2$ values, it was realised that it had the correct number of degrees of freedom to describe gravity as the $[a_1 \ldots a_{D-3}]$ would be equivalent to a single index.

As we have mentioned the field that occurs in the $E_{11} \otimes_s l_1$ non-linear realisation at level three in eleven dimensions has the indices $h_{a_1 \ldots a_8,b}$ and it was proposed that this would satisfy a duality relation with the usual gravity field that was first order in derivatives, generalising the duality between the three and six form fields found at levels one and two respectively [1]. Indeed an explicit equation of this type, generalised to D dimensions, and using the field $h_{a_1 \ldots a_{D-3},b}$ was given. It was shown that one could take a derivative of this duality relation so as to obtain an equation for the gravity field or the dual gravity field [1]. As a result it

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was shown that the dual gravity field $h_{a_1\ldots a_8,b}$ did indeed correctly propagate the degrees of freedom of gravity. In reference [10] it was shown that the formulation of dual gravity that resulted in five dimensions in [1] was the same as that given earlier in reference [8].

Although the gravity-dual gravity relations given in reference [1] did correctly propagate the gravity degrees of freedom this was at the linearised level and this relation was postulated rather than derived from the $E_{11}\otimes_s l_1$ non-linear realisation in that paper. Subsequently, a no go theorem was shown in reference [11] in which it was argued that one could not construct a theory that was equivalent to Einstein's theory using the non-linear graviton field.

In this paper, using the $E_{11}\otimes_s l_1$ non-linear realisation, we will find a fully non-linear, the second order in derivatives, equation of motion satisfied by the dual graviton. We will also find the equation, which is first order in derivatives and that relates the gravity field to the dual gravity field. By projecting the later relation we will recover the equation of motion of eleven dimensional supergravity for the gravity field, previously derived from the non-linear realisation, as well as the dual gravity field equation of motion already mentioned. We will derive these equations by carrying out the E_{11} variations of the previously derived equation, namely the six form equation of motion and the three form and six form duality equation respectively [4,5]. Thus, together with our previous results we have derived the full non-linear equations of motion from E theory up to and including level three, that is, the level containing the dual gravity field.

The dual graviton equation of motion that we will find involves both the dual graviton and the usual gravity field although the latter field is not present at the linearised level. As such it evades the no-go theorem of reference [11] as this assumes that the equation of motion of the dual gravity field just involves the dual gravity field. As such E_{11} has provided a way to evade the no-go theorem.

An account of the $E_{11}\otimes_s l_1$ non-linear realisation can be found in previous papers on E_{11}, for example in references [4,5], or the review of reference [6]. As is well known the fields of this theory, up to and including level three, are the graviton $h_a{}^b$, the three form field $A_{a_1a_2a_3}$, the six form field $A_{a_1\ldots a_6}$ and the dual graviton field $h_{a_1\ldots a_8,b}$ respectively. The dynamics is constructed from a group element $g\in E_{11}\otimes_s l_1$ using the Cartan forms $G_{\underline{\alpha}}$ and vielbein E^A which are defined by

$$g^{-1}dg = E^A l_A + G_{\underline{\alpha}} R^{\underline{\alpha}} \tag{1.1}$$

where $R^{\underline{\alpha}}$ and l_A are the generators of E_{11} and the vector representation l_1 respectively. The Cartan forms, up to level three, are given by [4,5]

$$\begin{aligned}
G_{\tau,a}{}^b &= (\det e)^{\frac12} e_a{}^\rho \partial_\tau e_\rho{}^b,\\
G_{\tau,\mu_1\mu_2\mu_3} &= (\det e)^{\frac12} e_{[a_1}{}^{\mu_1} e_{a_2}{}^{\mu_2} e_{a_3]}{}^{\mu_3}\partial_\tau A_{\mu_1\mu_2\mu_3},\\
G_{\tau,a_1\ldots a_6} &= (\det e)^{\frac12} e_{[a_1}{}^{\mu_1}\ldots e_{a_6]}{}^{\mu_6}\\
&\qquad\times(\partial_\tau A_{\mu_1\ldots\mu_6} - A_{\mu_1\mu_2\mu_3}\partial_\tau A_{\mu_4\mu_5\mu_6}),\\
G_{\tau,a_1\ldots a_8,b} &= (\det e)^{\frac12} e_{[a_1}{}^{\mu_1}\ldots e_{a_8]}{}^{\mu_8} e_b{}^\nu(\partial_\tau h_{\mu_1\ldots\mu_8,\nu}\\
&\qquad - A_{\mu_1\mu_2\mu_3}\partial_\tau A_{\mu_4\mu_5\mu_6}A_{\mu_7\mu_8\nu}\\
&\qquad + 2\partial_\tau A_{\mu_1\ldots\mu_6}A_{\mu_7\mu_8\nu}\\
&\qquad + 2\partial_\tau A_{\mu_1\ldots\mu_5\nu}A_{\mu_6\mu_7\mu_8})
\end{aligned} \tag{1.2}$$

where the vierbein is given in terms of the field $h_a{}^b$ by $e_\mu{}^a \equiv (e^h)_\mu{}^a$.

The advantage of working with the Cartan forms is that they only transform under the Cartan involution invariant subalgebra of E_{11}, denoted by $I_c(E_{11})$. At level zero this subalgebra is the Lorentz algebra and the variations at the next level have the parameter $\Lambda_{a_1a_2a_3}$ under which the Cartan forms transform as [4,5]

$$\delta G_a{}^b = 18\,\Lambda^{c_1c_2b}G_{c_1c_2a} - 2\,\delta_a^b\Lambda^{c_1c_2c_3}G_{c_1c_2c_3}, \tag{1.3}$$

$$\delta G_{a_1a_2a_3} = -\frac{5!}{2}G_{b_1b_2b_3a_1a_2a_3}\Lambda^{b_1b_2b_3} - 6\,G_{(c[a_1|)}\Lambda_{c|a_2a_3]} \tag{1.4}$$

$$\begin{aligned}
\delta G_{a_1\ldots a_6} &= 2\Lambda_{[a_1a_2a_3}G_{a_4a_5a_6]} - 112\,G_{b_1b_2b_3[a_1\ldots a_5,a_6]}\Lambda^{b_1b_2b_3}\\
&\quad + 112\,G_{b_1b_2a_1\ldots a_5a_6,b_3}\Lambda^{b_1b_2b_3}\\
&= 2\Lambda_{[a_1a_2a_3}G_{a_4a_5a_6]} - 336\,G_{b_1b_2b_3[a_1\ldots a_5,a_6]}\Lambda^{b_1b_2b_3}
\end{aligned} \tag{1.5}$$

$$\delta G_{a_1\ldots a_8,b} = -3\,G_{[a_1\ldots a_6}\Lambda_{a_7a_8]b} + 3\,G_{[a_1\ldots a_6}\Lambda_{a_7a_8b]} \tag{1.6}$$

The above formulae are true when the Cartan forms are written as forms, for example $G_a{}^b = dz^\Pi G_{\Pi,a}{}^b$. However, we will work with Cartan forms for which their first world volume index is converted into a tangent index, that is, $G_{A,\underline{\alpha}} = E_A{}^\Pi G_{\Pi,\underline{\alpha}}$. Under the $I_c(E_{11})$ transformations this index transforms

$$\delta G_{a,\bullet} = -3G^{b_1b_2}{}_{,\bullet}\,\Lambda_{b_1b_2a},\quad \delta G^{a_1a_2}{}_{,\bullet} = 6\Lambda^{a_1a_2b}G_{b,\bullet},\ \ldots \tag{1.7}$$

As such the Cartan forms $G_{A,\underline{\alpha}}$ transform under equations (1.3) to (1.6) on their second (E_{11}) index and under equation (1.7) on their first (l_1) index.

2. The non-linear dual graviton equation

In this section we will derive the non-linear equation for the dual graviton by carrying out the E_{11} variation of the six form field equation which is second order in derivatives. To derive this latter equation we begin with the duality relation between the three form and six form fields which is first order in derivatives and was derived from the $E_{11}\otimes_s l_1$ non-linear realisation [3–5]; it takes the familiar form

$$E_{\mu_1\ldots\mu_7} \equiv G_{[\mu_1,\ldots\mu_7]} + \frac{2}{7!}(\det e)^{-1}\epsilon_{\mu_1\ldots\mu_7}{}^{\nu_1\ldots\nu_4}G_{\nu_1,\nu_2\nu_3\nu_4} = 0 \tag{2.1}$$

In writing this equations we have suppressed the terms which contain derivatives with respect to the higher level coordinates; these terms can be found in references [3,4] and are also given in the next section.

Taking the derivative of equation (2.1) we find the equation

$$E^{\mu_1\ldots\mu_6} \equiv \partial_\nu\{(\det e)^{\frac12}G^{[\nu,\mu_1\ldots\mu_6]}\} = 0 \tag{2.2}$$

which is given in terms of objects with tangent indices by [3,4]

$$\begin{aligned}
E_{a_1\ldots a_6} &\equiv (\det e)^{\frac12}e^{d\mu}\partial_\mu G_{[d,a_1\ldots a_6]} + \frac12 G^d{}_{,b}{}^b G_{[d,a_1\ldots a_6]}\\
&\quad - G^{d,}{}^c{}_d G_{[c,a_1\ldots a_6]} - 6\,G^{d,}{}^c{}_{[a_1|}G_{[d,c|a_2\ldots a_6]]} = 0
\end{aligned} \tag{2.3}$$

Using the Cartan involution invariant ($I_c(E_{11})$) transformations of equations (1.3) to (1.6) we find that the variation of equation (2.3) leads to the expression.

$$\begin{aligned}
\delta E^{a_1\ldots a_6} &= -108\,G_{d_1,\,d_2c_1c_2}\,\Lambda^{c_1c_2[a_1|}\,G^{[d_1,\,d_2|a_2\ldots a_6]]}\\
&\quad + 12\,G_{d,\,c_1c_2c_3}\,\Lambda^{c_1c_2c_3}\,G^{[d,\,a_1\ldots a_6]}\\
&\quad - 18\,G_{c_1,\,c_2c_3d}\,\Lambda^{c_1c_2c_3}\,G^{[d,\,a_1\ldots a_6]}\\
&\quad + e_{\mu_1}{}^{[a_1}\ldots e_{\mu_6}{}^{a_6]}\partial_\nu\{(\det e)^{\frac12}2\Lambda^{[\mu_1\mu_2\mu_3}G^{\nu,\mu_4\mu_5\mu_6]}\\
&\quad + 168\,(\det e)^{\frac12}G^{[\nu,\mu_1\ldots\mu_6]\sigma_1\sigma_2,\sigma_3}\Lambda_{\sigma_1\sigma_2\sigma_3}\}.
\end{aligned} \tag{2.4}$$

In deriving equation (2.4) we have used equation of motion (2.1) and the previously derived [4,5] three form equation of motion; these terms will be reinstated when the variation is presented at the end of this section in equation (2.11).

Using the identity

$$\begin{aligned}
&2\,\Lambda^{c_1c_2c_3}\,\varepsilon^{a_1\ldots a_6 b_1\ldots b_5}\,S_{b_1\ldots b_4}\,S_{b_5c_1c_2c_3}\\
&\quad-9\,\Lambda^{c_1c_2[a_1}\,\varepsilon^{a_2\ldots a_6]b_1\ldots b_5 b_6}\,S_{b_1\ldots b_4}\,S_{b_5b_6c_1c_2}\\
&\quad=-\frac{5}{2}\,\Lambda^{[a_1a_2a_3}\,\varepsilon^{a_4a_5,a_6]b_1\ldots b_8}\,S_{b_1\ldots b_4}\,S_{b_5\ldots b_8},
\end{aligned}\tag{2.5}$$

valid for any $\Lambda_{a_1a_2a_3}$ and $S_{b_1\ldots b_4}$, both of which are totally anti-symmetric in all their indices, we find that the terms in equation (2.4) which are quadratic in the Cartan form $G_{a_1,a_2a_3a_4}$ vanish! The remaining terms result in the equation

$$\begin{aligned}
\delta E^{a_1\ldots a_6}&=168 e_{\mu_1}^{[a_1}\ldots e_{\mu_6}^{a_6]}\,\partial_\nu\left((\det e)^{\frac{1}{2}}\,G^{[\nu,\,\mu_1\ldots\mu_6]\sigma_1\sigma_2,\,\sigma_3}\,\Lambda_{\sigma_1\sigma_2\sigma_3}\right)\\
&=e_{\mu_1}{}^{a_1}\ldots e_{\mu_6}{}^{a_6}\\
&\quad\times\Big\{432\,\partial_{[\nu|}\{(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\sigma_2]}{}_{|\tau]}\}\Lambda_{\sigma_1\sigma_2}{}^{\tau}\\
&\quad+216\,\partial_\tau\{(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\sigma_2]}{}_{,\nu}\}\Lambda_{\sigma_1\sigma_2}{}^{\tau}\\
&\quad-48\,\partial_\nu\{(\det e)^{\frac{1}{2}}\,G^{\sigma_1,\sigma_2\nu\mu_1\ldots\mu_6}{}_{,\sigma_3}\Lambda_{\sigma_1\sigma_2}{}^{\sigma_3}\}\\
&\quad+864\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\lambda]}{}_{,\tau}\,G_{\nu,(\sigma_2\lambda)}\,\Lambda_{\sigma_1}{}^{\sigma_2\tau}\Big\}
\end{aligned}\tag{2.6}$$

As explained, for example in reference [4], the equation that results from the E_{11} variation is computed up to terms that only contain ordinary derivatives with respect to spacetime. However, to find this result we must add certain terms to the equation that is being varied, that is in this case equation (2.3), which contain derivatives with respect to the level one coordinate, $x^{a_1a_2}$. The reason for this is that any term of the form $f\,\partial_\tau g\,\Lambda^{\sigma_1\sigma_2\tau}$, where f and g are any function of the fields, whose indices we have suppressed, which appears in the variation can be removed by the addition of a term of the form $-6f\,\partial^{\sigma_1\sigma_2}g$ to the equation being varied. In order not to complicate the discussion we will list these additional terms at the end of this section.

We observe that the second and third terms in equation (2.6) are of the form just described in the paragraph above and so we can remove these by adding appropriate terms to the $E^{a_1\ldots a_6}$ equation of motion. The last term in equation (2.6) can be processed by using the gravity-dual gravity relation of equation (3.5), derived in the next section, to eliminate the dual gravity Cartan form in terms of the spin connection to find that

$$\begin{aligned}
&4\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\lambda]}{}_{,\tau}\,G_{\nu,(\sigma_2\lambda)}\,\Lambda_{\sigma_1}{}^{\sigma_2\tau}\\
&\quad=-2\,(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\lambda]}{}_{,\tau}\,(\omega_{\sigma_2,\nu\lambda}-Q_{\sigma_2,\nu\lambda})\Lambda_{\sigma_1}{}^{\sigma_2\tau}\\
&\quad=-\frac{4}{9!}\epsilon^{\mu_1\ldots\mu_6\sigma_1\kappa_1\kappa_2\rho_1\rho_2}\,\omega_{\sigma_3,\rho_1\rho_2}\,\omega_{\sigma_2,\kappa_1\kappa_2}\,\Lambda_{\sigma_1}{}^{\sigma_2\sigma_3}\\
&\qquad+2\,(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\lambda]}{}_{\tau}\,Q_{\sigma_2,\nu\lambda}\,\Lambda_{\sigma_1}{}^{\sigma_2\tau}
\end{aligned}\tag{2.7}$$

where $\omega_c{}^{ab}$ is the usual spin connection which is given by

$$\begin{aligned}
&(\det e)^{\frac{1}{2}}\,\omega_{c,ab}=-G_{a,(bc)}+G_{b,(ac)}+G_{c,[ab]},\quad\text{and}\\
&(\det e)^{\frac{1}{2}}\,Q_{a,bc}=G_{a,[bc]}
\end{aligned}\tag{2.8}$$

The first term in equation (2.7) vanishes due to symmetry arguments and the second terms can be removed adding terms with derivatives with respect to the level one coordinate to the equation being varied, that is, equation (2.3).

To derive the dual gravity equation of motion we must remove the $I_c(E_{11})$ parameter $\Lambda^{\sigma_1\sigma_2\sigma_3}$, however, as we have just discussed, we can add terms to the equation that are of the form $f\,\partial_\tau g\,\Lambda^{\sigma_1\sigma_2\tau}$ as long as we add the corresponding term that contain derivatives with respect to the level one coordinates to the six form equation. While such terms can be derived by using the E_{11} transformations we have fixed the coefficient of the one additional term in the dual gravity equation of motion by requiring that the equation be diffeomorphism invariant. We will give an alternative derivation of this term at the end of section four. The equation of motion for the dual graviton is then given by

$$\begin{aligned}
E^{\mu_1\ldots\mu_8}{}_{,\tau}&\equiv\partial_{[\nu|}\{(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_8]}{}_{,|\tau]}\}\\
&\quad+\frac{2}{9!}(\det e)^{-\frac{1}{2}}\,\epsilon^{\mu_1\ldots\mu_8\nu_1\nu_2\nu_3}\,G_{\tau,\nu_2}{}^{\rho}\,\omega_{\nu_1,\rho\nu_3}=0
\end{aligned}\tag{2.9}$$

The six form equation of motion with the additional terms mentioned above is given by

$$\begin{aligned}
\mathcal{E}^{a_1\ldots a_6}&=e_{\mu_1}{}^{[a_1}\ldots e_{\mu_6}{}^{a_6]}\{\partial_\nu\{(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6]}\}\\
&\quad-8\,\partial_\nu((\det e)^{\frac{1}{2}}\,G^{\tau_1\tau_2,\nu\mu_1\ldots\mu_6}{}_{\tau_1,\tau_2})\\
&\quad+\frac{1}{7}(\det e)^{-\frac{1}{2}}\,\partial^{\mu_1\mu_2}((\det e)^{\frac{1}{2}}\,G^{\mu_3,\mu_4\mu_5\mu_6})\\
&\quad-36\,e_{\tau_1}{}^{b_1}e_{\tau_2}{}^{b_2}e_{\rho_1b_1}e_{\rho_2b_2}\,\partial^{\rho_1\rho_2}\\
&\quad\times((\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6\tau_1\tau_2]}{}_{,\nu})\\
&\quad-72\,(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\ldots\mu_6\sigma_1\lambda]}{}_{,\tau}\,Q^{\tau}{}_{\sigma_1,\nu\lambda}\}\\
&\quad-3G^{c_1c_2}{}_{,c_1c_2e}\,G^{[e,a_1\ldots a_6]}-18\,G^{c[a_1|}{}_{,cd_1d_2}\,G^{[d_1,d_2|a_2\ldots a_6]]}\\
&\quad-\frac{2}{7!}\,(\det e)^{\frac{1}{2}}\,\epsilon^{a_1\ldots a_6c_1c_2c_3}{}_{d_1d_2}\,G^{d_1d_2,}{}_{c_1}{}^{d_3}\,\omega_{c_2,\,c_3d_3}.
\end{aligned}\tag{2.10}$$

and its final variation can be written as

$$\begin{aligned}
\delta\mathcal{E}^{a_1\ldots a_6}&=432\,\Lambda_{c_1c_2c_3}\,E^{a_1\ldots a_6c_1c_2,\,c_3}+\frac{8}{7}\Lambda^{[a_1a_2a_3}\,E^{a_4a_5a_6]}\\
&\quad+\frac{2}{105}G_{[e_5,c_1c_2c_3]}\epsilon^{a_1\ldots a_6e_1\ldots e_5}\,E_{e_1\ldots e_4}\,\Lambda^{c_1c_2c_3}\\
&\quad-\frac{3}{35}G_{[e_5,e_6c_1c_2]}\,\Lambda^{c_1c_2[a_1}\,\epsilon^{a_2\ldots a_6]e_1\ldots e_6}\,E_{e_1\ldots e_4}\\
&\quad+\frac{1}{420}\,\epsilon_{c_1}{}^{a_1\ldots a_6b_1\ldots b_4}\,\omega_{c_2,\,b_1b_2}\,E_{c_3,\,b_3b_4}\,\Lambda^{c_1c_2c_3}
\end{aligned}\tag{2.11}$$

where $E_{a,bc}$ is given in the next section in equation (3.5) and $E^{\mu_1\mu_2\mu_3}$ is the equation of motion of the three form gauge field which can be found by projecting equation (2.1) and it is given by [3–5]

$$\begin{aligned}
E^{\mu_1\mu_2\mu_3}&\equiv\partial_\nu\{(\det e)^{\frac{1}{2}}\,G^{[\nu,\mu_1\mu_2\mu_3]}\}\\
&\quad+\frac{1}{2.4!}(\det e)^{-1}\,\epsilon^{\mu_1\mu_2\mu_3\tau_1\ldots\tau_8}\,G_{[\tau_1,\tau_2\tau_3\tau_4]}\,G_{[\tau_5,\tau_6\tau_7\tau_8]}\\
&=0
\end{aligned}\tag{2.12}$$

We use the notation $\mathcal{E}$ rather than E when writing the symbols for the equations of motion that contain the terms with derivatives with respect to the level one coordinates.

3. The gravity-dual gravity relation

To find the gravity-dual gravity relation we will vary the three-six form duality relations given in equation (2.1) under the $I_c(E_{11})$ transformations given in equations (1.3) to (1.6). However, we now use the form of this relation when expressed in terms of four rather than seven antisymmetric indices and we include the terms which possess derivatives with respect to the level one coordinates;

$$\mathcal{E}_{a_1\dots a_4} \equiv \mathcal{G}_{a_1a_2a_3a_4} - \frac{1}{2.4!}\epsilon_{a_1a_2a_3a_4}{}^{b_1\dots b_7}\mathcal{G}_{b_1b_2\dots b_6,b_7} + \frac{1}{2}G_{[a_1a_2,a_3a_4]} \tag{3.1}$$

where

$$\mathcal{G}_{a_1a_2a_3a_4} \equiv G_{[a_1,a_2a_3a_4]} + \frac{15}{2}G^{b_1b_2}{}_{,b_1b_2a_1\dots a_4} \tag{3.2}$$

$$\mathcal{G}_{a_1a_2\dots a_7} \equiv G_{a_1,a_2\dots a_7} + 28\,G^{e_1e_2}{}_{,e_1e_2[a_1\dots,a_7]} \tag{3.3}$$

Varying under the $I_c(E_{11})$ transformations, and using the relation $\mathcal{E}^{(1)}{}_{a_1\dots a_4} = 0$, we find that

$$\begin{aligned} & e^{\mu_1}_{[a_1}\dots e^{\mu_4}_{a_4]}\delta\mathcal{E}^{a_1\dots a_4} \\ &= 3(\det e)^{\frac{1}{2}}\omega_{\tau,}{}^{\mu_1\mu_2}\Lambda^{\mu_3\mu_4\tau} - \frac{7}{2}(\det e)^{-1}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7} \\ &\quad\times (G_{\nu_1,\nu_2\dots\nu_7\tau_1\tau_2,\tau_3} + G_{\tau_1,\tau_2\tau_3\nu_1\nu_2\dots\nu_6,\nu_7})\Lambda^{\tau_1\tau_2\tau_3} \\ &= 0 \end{aligned} \tag{3.4}$$

Extracting off $\Lambda^{\rho_1\rho_2\rho_3}$ and setting on $\mu_3 = \rho_2$ and $\mu_4 = \rho_3$ and summing in equation (3.4) we find, after scaling by $(\det e)^{-\frac{1}{2}}$, and converting to worldvolume indices, that

$$E_{\tau,}{}^{\mu_1\mu_2} \equiv (\det e)\omega_{\tau,}{}^{\mu_1\mu_2} - \frac{1}{4}(\det e)^{-\frac{1}{2}}\epsilon^{\mu_1\mu_2\nu_1\dots\nu_9}G_{\nu_1,\nu_2\dots\nu_9,\tau} \doteq 0$$

or equivalently

$$E^{\mu_1\mu_2\dots\mu_9}{}_{,\tau} \equiv (\det e)^{\frac{1}{2}}G^{[\mu_1,\mu_2\dots\mu_9]}{}_{,\tau} + \frac{2}{9!}\epsilon^{\mu_1\mu_2\dots\mu_9\nu_1\nu_2}\omega_{\tau,\nu_1\nu_2} \tag{3.5}$$

To find this equation we have used the identity

$$3X_{[a_1a_2}{}^{[c_1}\delta^{c_2c_3]}_{a_3a_4]}\delta^{a_3}_{c_2}\delta^{a_4}_{c_3} = \frac{14}{3}X_{a_1a_2}{}^{c_1} - \frac{8}{3}X_{d[a_1}{}^{d}\delta^{c_1}_{a_2]} \tag{3.6}$$

for any tensor $X_{a_1a_2}{}^{c}$ which obeys $X_{a_1a_2}{}^{c} = X_{[a_1a_2]}{}^{c}$.

As we have explained previously [12,5,7,14] equation (3.5) only holds modulo certain transformations. Clearly one of these is the familiar local Lorentz transformations which are the level zero parts of $I_c(E_{11})$. However, there are also parts of the level three gauge transformations of the field $h_{a_1\dots a_8,b}$ and as we will discuss later in this section the gauge transformations of the three and six form gauge fields. The use of the symbol $\doteq$ signifies that the equation is to be understood in this way.

By tracing equation (3.5) we find the relation

$$\omega_{\rho,}{}^{\rho\mu} \doteq 0 \tag{3.7}$$

Indeed we have already used this equation in the derivation of equation (3.5). Clearly if this equation were to hold exactly , rather than only holding modulo the above mentioned transformations, it would be incompatible with the correct propagation of a spin two field. However, as has been explained in detail in references [12,5,14] once this point is taken into account equation (3.7) is completely compatible with Einstein gravity. A gravity dual gravity relation similar to that of equation (3.5) was given in reference [3]. However, the modulo nature to the equation was only alluded to and the projection to find the gravity equation was not understood at that time and as a result it was not claimed that this was the final version of the gravity-dual gravity equation.

Given the more complicated index structure of equation (3.4) compared to equation (3.5) it is far from clear that taking the double trace of equation (3.4) leads to the full content of equation (3.5) and that there are not further equations which might not be compatible with the propagation of the degrees of freedom of gravity. In fact this is not the case, the second term on the right hand side of equation (3.4) can be processed as following

$$\begin{aligned} & -\frac{7}{2}\,(\det e)^{-\frac{1}{2}}\,\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7} \\ &\quad\times \left(G_{\nu_1,\,\nu_2\dots\nu_7\tau_1\tau_2,\,\tau_3} + G_{\tau_1,\,\tau_2\tau_3\nu_1\dots\nu_6,\,\nu_7}\right)\Lambda^{\tau_1\tau_2\tau_3} \\ &= -\frac{7}{2}\,(\det e)^{-\frac{1}{2}}\,\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7} \\ &\quad\times\left(G_{\nu_1,\,\nu_2\dots\nu_7\tau_1\tau_2,\,\tau_3} + \frac{2}{7}\,G_{\tau_1,\,\tau_2\nu_1\dots\nu_7,\,\tau_3}\right)\Lambda^{\tau_1\tau_2\tau_3} \end{aligned} \tag{3.8}$$

In deriving this equation we have used the SL(11) irreducibility of the dual gravity Cartan form, that is $G_{\tau,\,[\nu_1\dots\nu_8,\lambda]} = 0$. We can now combine the two terms in equation (3.8) together to find the expression

$$\begin{aligned} & -\frac{1}{2}\,(\det e)^{-\frac{1}{2}}\,\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7} \\ &\quad\times\left(7\,G_{\nu_1,\,\nu_2\dots\nu_7\tau_1\tau_2,\,\tau_3} + 2\,G_{\tau_1,\,\tau_2\nu_1\dots\nu_7,\,\tau_3}\right)\Lambda^{\tau_1\tau_2\tau_3} \\ &= -\frac{9}{2}\,(\det e)^{-\frac{1}{2}}\,\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7}G_{[\nu_1,\,\nu_2\dots\nu_7\tau_1\tau_2],\,\tau_3}\,\Lambda^{\tau_1\tau_2\tau_3} \end{aligned} \tag{3.9}$$

Finally, by using the identity $-\frac{1}{2.9!}\epsilon_{\lambda_1\lambda_2\rho_1\dots\nu_9}\epsilon^{\lambda_1\lambda_2\sigma_1\dots\sigma_9} = \delta^{\sigma_1\dots\sigma_9}_{\rho_1\dots\rho_9}$ in this last equation one finds the expression

$$\begin{aligned} & \frac{1}{4.8!}\,(\det e)^{-\frac{1}{2}}\,\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7}\epsilon_{\lambda_1\lambda_2\tau_1\tau_2\nu_1\dots\nu_7}\epsilon^{\lambda_1\lambda_2\sigma_1\dots\sigma_9} \\ &\quad\times G_{\sigma_1,\,\sigma_2\dots\sigma_9,\,\tau_3}\,\Lambda^{\tau_1\tau_2\tau_3} \\ &= -\frac{3}{4}\,(\det e)^{-\frac{1}{2}}\,\Lambda^{\tau[\mu_1\mu_2}\epsilon^{\mu_3\mu_4]\sigma_1\dots\sigma_9}G_{\sigma_1,\,\sigma_2\dots\sigma_9,\,\tau}\,. \end{aligned} \tag{3.10}$$

The result is that equation (3.4), once we have reinstated the three form-six form relation, can be written as

$$\delta\mathcal{E}^{\,a_1\dots a_4} = \frac{1}{4!}\,\varepsilon^{a_1\dots a_4b_1\dots b_7}\,E_{b_1\dots b_4}\,\Lambda_{b_5b_6b_7} + 3(\det e)^{-\frac{1}{2}}\,E^{[a_1a_2}_{c}\,\Lambda^{a_3a_4]c}. \tag{3.11}$$

From this equation it is clear that the E_{11} variation of the three form-six form relation leads to the same four form equation and the gravity-dual gravity relation of equation (3.5) and no other constraints. The derivation given here corrects a mistake in reference [7] in which the equation (4.5) is incorrect as one of the terms was substituted by the wrong expression when writing up the paper.

4. Derivation of Einstein equation from the gravity-dual gravity relation

In this section we will project the gravity-dual gravity relation of equation (3.5), which is first order in derivatives, to find two equations that are second order in derivatives, one of which contains only the graviton field and the other of which contains only the dual graviton. We must also do this in such a way that the resulting equation holds exactly and so the projection must also eliminate the modular transformations which the gravity-dual gravity relation possess. We begin by considering the expression

$$\partial_\nu(E_{\tau,}{}^{\nu\mu}) - \det e\;\partial_\tau\{(\det e)^{-1}E_{\nu,}{}^{ab}\}e_a{}^{\mu}e_b{}^{\nu} \tag{4.1}$$

where $E_{\tau,}{}^{\mu\nu}$ is the gravity-dual gravity relation of equation (3.5). This expression would vanish if we forgot that $E_{\tau,}{}^{\mu\nu}$ only hold modulo certain transformations. The parts of equation (4.1) that depend on ω_τ^{ab} are given by

$$\partial_\nu(\det e\,\omega_{\tau,}{}^{\nu\mu}) - \det e\,\partial_\tau\omega_{\nu,}{}^{ab}e_a{}^\mu e_b{}^\nu = \det e R_\tau{}^\mu \tag{4.2}$$

The reader may verify that $R_\tau{}^a$ is the same as the well known expression for the Ricci tensor. Of course it transforms covariantly under local Lorentz and diffeomorphism transformations.

The part of the first term of equation (4.1) that contains the dual graviton Cartan form is of the form

$$-\frac{1}{4}\epsilon^{\rho\mu\nu_1\ldots\nu_9}\partial_\rho((\det e)^{-\frac{1}{2}}G_{\nu_1,\nu_2\ldots\nu_9,\tau}) \tag{4.3}$$

We recognise that the expression on the right-hand side of this equation contains the Bianchi identity for the dual graviton Cartan form which can be evaluated by considering the explicit for the Cartan form given in equation (1.2), or by using the E_{11} Maurer–Cartan equation, the result is

$$\begin{aligned}&\partial_{[\rho}((\det e)^{-\frac{1}{2}}G_{\nu_1,\nu_2\ldots\nu_9],\tau})\\&\quad=2\,(\det e)^{-1}(G_{[\nu_1,\nu_2\ldots\nu_7}G_{\rho,\nu_8\nu_9]\tau}+G_{[\nu_1,\nu_2\ldots\nu_6|\tau|}G_{\rho,\nu_7\nu_8\nu_9]})\end{aligned} \tag{4.4}$$

The part of the second term of equation (4.1) that involves the dual graviton Cartan form is then given by

$$\frac{1}{4}(\det e)^{-\frac{1}{2}}\partial_\tau e_\rho{}^b e_b{}^\lambda\epsilon^{\mu\rho\nu_1\ldots\nu_9}G_{\nu_1,\nu_2\ldots\nu_9,\lambda} \tag{4.5}$$

Using equations (4.4) and (4.5) we then find that

$$\begin{aligned}&\partial_\nu(E_\tau{}^{\nu\mu}) - \det e\,\partial_\tau\{(\det e)^{-1}E_{\nu,}{}^{ab}\}e_a{}^\mu e_b{}^\nu\\&\quad=(\det e)R_\tau{}^\mu - 4(12G^{[\mu,\rho_1\ldots\rho_3]}G_{[\tau,\rho_1\ldots\rho_3]}\\&\qquad-\delta_\tau^\mu G^{[\rho_1,\rho_2\rho_3\rho_4]}G_{[\rho_1,\rho_2\rho_3\rho_4]})\\&\qquad+\frac{1}{4}\epsilon^{\rho\mu\nu_1\ldots\nu_9}\{\frac{2}{3}G_{\rho,\nu_1\ldots\nu_6}G_{\tau,\nu_7\nu_8\nu_9}+\frac{1}{3}G_{\tau,\nu_1\ldots\nu_6}G_{\rho,\nu_7\nu_8\nu_9}\}\\&\qquad+\frac{1}{4}\partial_\tau e_\rho{}^b e_b{}^\lambda\epsilon^{\mu\rho\nu_1\ldots\nu_9}G_{\nu_1,\nu_2\ldots\nu_9,\lambda}\end{aligned} \tag{4.6}$$

In the second term of this last equation we recognise the energy momentum tensor of the eleven dimensional supergravity theory.

In equation (4.1) we are taking the derivative of Cartan forms and as these transform in a non-trivial way one would normally have to take a covariantised derivative. However, if one were taking the derivative of objects that vanish then clearly no covariantisation would be required as one would be constructing an equation which is automatically true, or put another way, the required covariantisation vanishes. This is the situation that occurs when we take the derivatives of the three form-six form relation of equation (2.1) to find the equation of motion for these fields. However, the situation we encounter in equation (4.1) is different, and indeed rather subtle, as the gravity-dual gravity relation of equation (3.5) does vanish except for the fact that it only holds modulo certain transformations. The projection we have taken is covariant with respect to local Lorentz transformations and we now compute the two and five form gauge transformations of the dual gravity Cartan form to find what covariantisations we must add. We take the three form and six form gauge fields to have the transformations

$$\begin{aligned}&\delta e_\mu{}^a=0,\quad \delta A_{\mu_1\mu_2\mu_3}=\partial_{[\mu_1}\Lambda_{\mu_2\mu_3]},\\&\delta A_{\mu_1\ldots\mu_6}=\partial_{[\mu_1}\Lambda_{\mu_2\ldots\mu_6]}+\partial_{[\mu_1}\Lambda_{\mu_2\mu_3}A_{\mu_3\ldots\mu_6]}\end{aligned} \tag{4.7}$$

These transformations are the well known gauge transformations for these fields of eleven dimensional supergravity and they leave invariant the seven form field strength $G_{[\mu_1,\mu_2\ldots\mu_7]}$. They can also be deduced from the E_{11} motivated gauge transformations of reference [13]. Clearly the spin connection is invariant under theses gauge transformations. However, this is not the case for the Cartan form associated with the dual graviton field as it contains terms non-linear in the gauge fields. Using equation (4.4) we find that the object that occurs in equation (4.3) transforms under these gauge transformations as

$$\begin{aligned}&\delta(\partial_{[\mu_1}((\det e)^{-\frac{1}{2}}G_{\mu_2,\mu_3\ldots\mu_{10}],\tau}))\\&\quad=-\frac{2}{3}\partial_{[\mu_1}A_{\mu_2\ldots\mu_7}\partial_{|\tau|}\partial_{\mu_8}\Lambda_{\mu_9\mu_{10}]}\\&\qquad-\frac{1}{3}\partial_\tau\partial_{[\mu_1}\Lambda_{\mu_2\ldots\mu_6}\partial_{\mu_7}A_{\mu_8\ldots\mu_{10}]}\end{aligned} \tag{4.8}$$

As a result to ensure that the projection of equation (4.6) is covariant with respect to modulo transformations we must add to the left hand side of equation (4.6) the term

$$-\frac{1}{4}\epsilon^{\nu_1\mu\nu_2\ldots\nu_{10}}\{\frac{2}{3}G_{\nu_1,\nu_2\ldots\nu_7}G_{\tau,\nu_8\nu_9\nu_{10}}+\frac{1}{3}G_{\tau,\nu_1\ldots\nu_6}G_{\nu_7,\nu_8\nu_9\nu_{10}}\} \tag{4.9}$$

The effect of adding this term is to exactly cancel the term quadratic in $G_{a_1,a_2\ldots a_4}$ in the second line on the right hand side of equation (4.6).

Proceeding in a similar way for diffeomorphism transformations one finds that one must add to the left hand side of equation (4.6) the term

$$-\frac{1}{4}\partial_\tau e_\rho{}^b e_b{}^\lambda\epsilon^{\mu\rho\nu_1\ldots\nu_9}G_{\nu_1,\nu_2\ldots\nu_9,\lambda} \tag{4.10}$$

so cancelling the last term on the right hand side of equation (4.6).

The effect of all these steps is the equation

$$\begin{aligned}&\partial_\nu(E_{\tau,}{}^{\nu\mu})-\det e\,\partial_\tau\{(\det e)^{-1})E_{\nu,}{}^{ab}\}e_a{}^\mu e_b{}^\nu\\&\quad-\frac{1}{4}\partial_\tau e_\rho{}^b e_b{}^\lambda\epsilon^{\mu\rho\nu_1\ldots\nu_9}G_{\nu_1,\nu_2\ldots\nu_9,\lambda}\\&\quad-\frac{1}{4}\epsilon^{\nu_1\mu\nu_2\ldots\nu_{10}}\{\frac{2}{3}G_{\nu_1,\nu_2\ldots\nu_7}G_{\tau,\nu_8\nu_9\nu_{10}}+\frac{1}{3}G_{\tau,\nu_1\ldots\nu_6}G_{\nu_7,\nu_8\nu_9\nu_{10}}\}\\&=0=(\det e)R_\tau{}^\mu-4(12\,G^{[\mu,\rho_1\ldots\rho_3]}G_{[\tau,\rho_1\ldots\rho_3]}\\&\quad-\delta_\tau^\mu G^{[\rho_1,\rho_2\rho_3\rho_4]}G_{[\rho_1,\rho_2\rho_3\rho_4]})\end{aligned} \tag{4.11}$$

We recognise the last equation as the equation of motion for gravity of eleven dimensional supergravity. This equation has previously been derived as following from the $E_{11}\otimes_s l_1$ non-linear realisation but by varying the three form equation of motion, which is second order in derivatives, under $I_c(E_{11})$ transformations [4,5]. In this later way of proceeding one does not encounter any equations that are modulo and as a result it confirms that the covariantisation of modulo transformations used above is consistent with E_{11}.

We now project the gravity-dual gravity equation in such a way that the gravity field drops out leaving the dual gravity field. For this we consider the dual graviton kinetic term and evaluate it in terms of the spin connection using the gravity-dual gravity relation of equation (3.5)

$$\begin{aligned}&\partial_{[\nu|}G^{[\nu,\mu_1\ldots\mu_8]}{}_{,|\tau]}\\&\quad=-\frac{2}{9!}\epsilon^{\mu_1\ldots\mu_8\nu\rho_1\rho_2}(\partial_{[\nu}\omega_{\tau],a_1a_2}+\omega_{[\nu|,ca_1}\omega_{|\tau],ca_2})e_{\rho_1}{}^{a_1}e_{\rho_2}{}^{a_2}\\&\qquad+\epsilon^{\mu_1\ldots\mu_8\nu_1\nu_2\nu_3}G_{\tau,\nu_2}{}^a\omega_{\nu_1,a\nu_3}\end{aligned}$$

$$= -\frac{2}{9!}\{\epsilon^{\mu_1\ldots\mu_8\nu\rho_1\rho_2} R_{\nu\tau,\rho_1\rho_2} + \epsilon^{\mu_1\ldots\mu_8\nu_1\nu_2\nu_3} G_{\tau,\nu_2}{}^a \omega_{\nu_1,a\nu_3}\}$$
$$= -\frac{2}{9!}(\det e)^{-\frac{1}{2}} \epsilon^{\mu_1\ldots\mu_8\nu_1\nu_2\nu_3} G_{\tau,\nu_2}{}^a \omega_{\nu_1,a\nu_3} \quad (4.12)$$

We have found that the Riemann tensor appears in the form of its Bianchi identity and so this term vanishes. Examining the final result we recognise that we have derived the dual graviton equation (2.9). In finding the dual graviton equation in this way we have given an alternative derivation of the second term in equation (2.9) which was previously derived using diffeomorphism arguments. This latter term is then part of the above covariantisation of the modular transformations discussed earlier. It would be good to understand this mechanism in a more conceptual manner.

As a result we have shown that the gravity-dual gravity relation of equation (3.5) implies both the gravity equation of motion of eleven dimensional supergravity as well as the corresponding equation for the dual graviton.

Discussion

In this paper we have completed the calculation of the fully non-linear equations of motion that follow from the $E_{11} \otimes_s l_1$ non-linear realisation, that is E theory, up to and including level three which contains the dual gravity field. When the equations are truncated to contain the graviton, three form, six form and dual gravity fields and the usual eleven dimensional space time they describe precisely eleven dimensional supergravity. This, together with the linearised results up to level four, should leave the reader in no doubt that the conjecture that E theory contains eleven dimensional supergravity, given in references [1,2] has been demonstrated. The equations of motion follow essentially uniquely from the Dynkin diagram of E_{11}.

Using E_{11} variations, we also found the gravity and dual gravity relation which is first order in derivatives. We also showed that this duality equation implies the equations of motion for both of these fields that are second order in derivatives.

E theory is a unified theory in that it contains not only eleven dimensional supergravity but also all the maximal supergravities, depending what decomposition one takes, and also all the gauged maximal supergravities, depending which fields one turns on, see reference [6] for a review. As such E theory should be the low energy effective action for type II strings and branes replacing the maximal supergravities in this role. There are many more coordinates and fields in E theory beyond those found in supergravity and it would be good to systematically understand what their physical meaning is.

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References

[1] P. West, E_{11} and M Theory, Class. Quant. Grav. 18 (2001) 4443, arXiv:hep-th/0104081.
[2] P. West, E_{11}, SL(32) and central charges, Phys. Lett. B 575 (2003) 333–342, arXiv:hep-th/0307098.
[3] P. West, Generalised geometry, eleven dimensions and E_{11}, J. High Energy Phys. 1202 (2012) 018, arXiv:1111.1642.
[4] A. Tumanov, P. West, E11 must be a symmetry of strings and branes, arXiv:1512.01644.
[5] A. Tumanov, P. West, E11 in 11D, Phys. Lett. B 758 (2016) 278, arXiv:1601.03974.
[6] P. West, Introduction to Strings and Branes, Cambridge University Press, 2012.
[7] A. Tumanov, P. West, E_{11}, Romans theory and higher level duality relations, Int. J. Mod. Phys. A 32 (26) (2017) 1750023, arXiv:1611.03369.
[8] T. Curtright, Generalised gauge fields, Phys. Lett. B 165 (1985) 304.
[9] C. Hull, Strongly coupled gravity and duality, Nucl. Phys. B 583 (2000) 237, arXiv:hep-th/0004195.
[10] N. Boulanger, S. Cnockaert, M. Henneaux, A note on spin-s duality, J. High Energy Phys. 0306 (2003) 060, arXiv:hep-th/0306023.
[11] X. Bekaert, N. Boulanger, M. Henneaux, Consistent deformations of dual formulations of linearized gravity: a no-go result, Phys. Rev. D 67 (2003) 044010, arXiv:hep-th/0210278;
X. Bekaert, N. Boulanger, S. Cnockaert, No self-interaction for two-column massless fields, J. Math. Phys. 46 (2005) 012303, arXiv:hep-th/0407102.
[12] P. West, Dual gravity and E11, arXiv:1411.0920.
[13] P. West, Generalised space–time and gauge transformations, J. High Energy Phys. 1408 (2014) 050, arXiv:1403.6395.
[14] P. West, On the different formulations of the E11 equations of motion, Mod. Phys. Lett. A 32 (18) (2017) 1750096, arXiv:1704.00580.